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Cerenkov Radiation in Anisotropic Media by the Methods of Quantum Electrodynamics

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We examine the Cerenkov effect in a transparent anisotropic medium by the methods of quantum electrodynamics. We first show that in such a medium the electric field intensity is not transverse. By resolving the field we distinguish the contributions to the intensity of the radiation corresponding to the transverse and longitudinal components of the field. Focusing also on the role of the spin, we show that its effect is significant since the intensity can increase or decrease compared with that of a spinless particle.

1. INTRODUCTION

Before 1934, it was generally accepted that only accelerating particles can radiate. Indeed there was some argument in scientific journals as to whether a particle which is accelerating uniformly can radiate (see, for instance, Born, 1909; Schott, 1915; Drukey, 1949; Bondi and Gold, 1955; Fulton and Rohrlich, 1966). Cerenkov's (1934, 1936) discovery in that year that a nonaccelerating particle can radiate therefore aroused great interest. It was not long before Tamm and Frank (1937) came up with an explanation, based on theoretical considerations, that a charged particle moving in a medium even with uniform velocity can radiate provided its velocity is greater than the phase velocity of light in that medium, that is,

$$v > \frac{c}{n} \tag{1.1}$$

The relevant computations were carried out by the methods of classical electrodynamics. This was followed by extensive studies on several aspects of the phenomenon. The methods employed included quantum electrodynam-

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ics. The first theoretical physicists to use quantum electrodynamics were V. L. Ginsburg and A. A. Sokolov. Using a phenomenological approach to account for the quantum properties of the medium, Sokolov (1940) obtained for the angle of radiation the expression

$$\cos\theta = \frac{1}{\beta n} + \frac{n\omega\hbar}{2cp}(1 - n^{-2}) \tag{1.2}$$

where $\beta = v/c$ and n is the refractive index of the medium. For the intensity of radiation he obtained

$$W = \frac{e^2}{c}\beta \int \omega\, d\omega \left[1 - \frac{1}{\beta^2 n^2} - \frac{\omega\hbar}{cp\beta}(1 - n^{-2}) + \frac{n^2\omega^2\hbar^2}{4c^2p^2}(1 - n^{-4})\right. \tag{1.3}$$

It is easy to see that when $\hbar \to 0$ we recover the expressions obtained by Tamm and Frank by the classical approach. Other investigations which used the methods of quantum electrodynamics include Taniuti (1951), Tidman (1956), Neatman (1953), Schiff (1955), and Tauch and Watson (1948, 1949).

Loskutov (1960) examined the relation between the spin of a particle and the polarization of its radiation. In all the above cases, the media considered were isotropic.

As in the case of the classical approach, quantum methods extended to other types of media: anisotropic, uniaxial crystals.

One distinctive feature of anisotropic media compared with isotropic media is that in the former case, the field of radiation is not transverse. This would imply that not all the energy radiated should be attributed to the Cerenkov effect. A similar picture may be referred to in the case of plasma, where because of anisotropy, longitudinal oscillations occur. The objective of this paper, being an extension of a previous one (Kukanov and Orisa, 1971), is to effect a separation of the contributions to the radiation between the transverse and longitudinal components. We shall also examine conditions for the existence of the radiation and obtain for the case of anisotropic media, conditions similar to those obtained by Tamm and Frank for the isotropic medium.

One advantage of resolving the field into transverse and longitudinal components as is being contemplated here is that this will make any study of the polarization of the radiation easier. Lastly, we shall consider a particle possessing spin, in order to pay attention to the role of the spin in the process of radiation.

The Cerenkov effect has found application in technology and in particular in the physics of high energy [(Jelly, 1960; Zrelov, 1964, 1968)]. These applications are based on the classical results. It is hoped that studies such as the present one can lead to applications where the focus will be on the quantum effects.

In the next section, we shall consider the characteristics of the field of radiation in anisotropic media. The third section is devoted to the computation of the energy loss. The results are expressed in terms of elliptic integrals. the paper then concludes with an examination of the results. We also show here that previous results for the isotropic medium can be recovered by an appropriate limit process.

2. QUANTIZATION OF THE ELECTROMAGNETIC FIELD IN A MEDIUM

In a medium, Maxwell's equations take the form

$$e_{\alpha\beta\gamma}\frac{\partial H_\gamma}{\partial X_\beta} = \frac{1}{c}\frac{\partial D_\alpha}{\partial t}, \qquad e_{\alpha\beta\gamma}\frac{\partial E_\gamma}{\partial X_\beta} = -\frac{1}{c}\frac{\partial B_\alpha}{\partial t}$$

$$\frac{\partial D_\alpha}{\partial X_\alpha} = \frac{\partial B_\alpha}{\partial X_\alpha} = 0$$

The electric and magnetic inductions are connected with the fields by the relations

$$D_\alpha = \varepsilon_{\alpha\beta}E_\beta, \qquad B_\alpha = \mu_{\alpha\beta}H_\beta$$

where $\varepsilon_{\alpha\beta}$ and $\mu_{\alpha\beta}$ are given by

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}; \qquad \mu_{\alpha\beta} = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix} \tag{2.1}$$

For a nondispersive medium, $\varepsilon_{\alpha\beta}$ and $\mu_{\alpha\beta}$ are constant. We choose a gauge in which the scalar potential is zero, so that

$$E_\alpha = -\frac{1}{c}\frac{\partial A_\alpha}{\partial t}, \qquad B_\alpha = e_{\alpha\beta\gamma}\frac{\partial A_\gamma}{\partial X_\beta}$$

Then the vector potential A_α satisfies the equation

$$\left(e_{\alpha\beta\gamma}\mu^{-1}_{\gamma\lambda}e_{\lambda\mu\nu}\frac{\partial^2}{\partial X_\beta \partial X_\mu} + \frac{1}{c^2}\varepsilon_{\alpha\beta}\frac{\partial^2}{\partial t^2}\right)A_\nu = 0 \tag{2.2}$$

with the supplementary condition

$$\varepsilon_{\alpha\beta}\frac{\partial A_\beta}{\partial X_\alpha} = 0 \tag{2.3}$$

It is usual to make a Fourier expansion of the vector potential. Alexeev and Nikitin (1965) have generalized the methods of Sokolov and Ginsburg for the isotropic medium to the case of anisotropic media. Making use of both ideas, we express the solution of equation (2.2) in the form

$$\mathbf{A} = L^{-3/2} \sum \left(\frac{4\pi c^2 \hbar}{\mathcal{M}} \right)^{1/2} (\mathbf{C} e^{-i\omega t + i\mathbf{q}\cdot\mathbf{r}} + \mathbf{C}^{+} e^{i\omega t - i\mathbf{q}\cdot\mathbf{r}}) \tag{2.4}$$

where L is the dimension of an arbitrary elementary cube and the amplitude $\mathbf{C}$ consists of classical and quantum parts. We may thus write

$$\mathbf{C} = \mathscr{f} \boldsymbol{l}$$

where $\boldsymbol{l}$ is the classical part such that

$$\mathcal{M}_j = \frac{d(\omega^2 \varepsilon_{\alpha\beta})}{d\omega} l_{j\alpha} l_{j\beta}^{+}$$

and the quantum part $\mathscr{f}$ satisfies the commutation relation

$$\mathscr{f}_{q,j} \mathscr{f}_{q',j'}^{+} - \mathscr{f}_{q',j'}^{+} \mathscr{f}_{q,j} = \delta_{qq'} \delta_{jj'}$$

Assuming that initially there are no photons in the field, we can put

$$\mathscr{f}_{q,j} \mathscr{f}_{q'j'}^{+} = \delta_{qq'} \delta_{jj'}$$

$$\mathscr{f}_{q'j'}^{+} \mathscr{f}_{q,j} = 0$$

The system (2.2) has nontrivial solutions if

$$D(n^2 - n_{+1}^2)(n^2 - n_{-1}^2) = 0$$

where

$$D = \mu_1^{-2} \mu_3^{-1} (\varepsilon_1 \sin^2\theta + \varepsilon_3 \cos^2\theta)(\mu_1 \sin^2\theta + \mu_3 \cos^2\theta) \neq 0$$

This gives two real values of n corresponding to two types of radiated waves, given by

$$n^2 = n_{+1}^2 = n_{\mu}^2 = \frac{\varepsilon_1 \mu_1 \mu_3}{\mu_1 \sin^2\theta + \mu_3 \cos^2\theta}$$

$$n^2 = n_{-1}^2 = n_{\varepsilon}^2 = \frac{\mu_1 \varepsilon_1 \varepsilon_3}{\varepsilon_1 \sin^2\theta + \varepsilon_3 \cos^2\theta}$$

connected with the wave vector by

$$q_j = n_j \frac{\omega}{c} \qquad (j = \mu, \varepsilon)$$

The corresponding normalized solutions become

$$\boldsymbol{l}_\mu = (\sin\varphi, -\cos\varphi, 0) \tag{2.5}$$

$$\boldsymbol{l}_\varepsilon = \frac{(\varepsilon_3\cos\theta\cos\varphi, \varepsilon_3\cos\theta\sin\varphi, -\varepsilon_1\sin\theta)}{(\varepsilon_1^2\sin^2\theta + \varepsilon_3^2\cos^2\theta)^{1/2}} \tag{2.6}$$

We represent the unit vector in the direction of the wave vector as

$$\frac{\mathbf{q}}{q} = \boldsymbol{\kappa} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \tag{2.7}$$

We then note that

$$\boldsymbol{l}_\mu\cdot\boldsymbol{l}_\varepsilon = \boldsymbol{l}_\mu\cdot\boldsymbol{\kappa} = 0$$

$$\boldsymbol{l}_\varepsilon\cdot\boldsymbol{\kappa} = \frac{(\varepsilon_3 - \varepsilon_1)\sin\theta\cos\theta}{(\varepsilon_1^2\sin^2\theta + \varepsilon_3^2\cos^2\theta)^{1/2}} \neq 0$$

Thus the vector $\boldsymbol{l}_\varepsilon$ is not transverse. To resolve $\boldsymbol{l}_\varepsilon$ into transverse and longitudinal components, we note that the vectors

$$\frac{\boldsymbol{\kappa}\wedge\mathbf{k}^0}{[1-(\boldsymbol{\kappa}\cdot\mathbf{k}^0)^2]^{1/2}} = \mathbf{e}_1; \qquad \frac{\boldsymbol{\kappa}\wedge(\boldsymbol{\kappa}\wedge\mathbf{k}^0)}{[1-(\boldsymbol{\kappa}\cdot\mathbf{k}^0)^2]^{1/2}} = \mathbf{e}_2; \qquad \boldsymbol{\kappa}$$

form a system of mutually orthonormal vectors; $\mathbf{k}^0$ is the unit vector along the optical axis.

We then resolve $\boldsymbol{l}_\varepsilon$ so that

$$\boldsymbol{l}_\varepsilon = (\boldsymbol{l}_\varepsilon\cdot\mathbf{e}_1)\mathbf{e}_1 + (\boldsymbol{l}_\varepsilon\cdot\mathbf{e}_2)\mathbf{e}_2 + (\boldsymbol{l}_\varepsilon\cdot\boldsymbol{\kappa})\boldsymbol{\kappa}$$

where

$$(\boldsymbol{l}_\varepsilon, \boldsymbol{\kappa}) = \frac{(\varepsilon_3 - \varepsilon_1)\sin\theta\cos\theta}{(\varepsilon_1^2\sin^2\theta + \varepsilon_3^2\cos^2\theta)^{1/2}} \tag{2.8}$$

$$(\boldsymbol{l}_\varepsilon\cdot\mathbf{e}_1) = 0$$

$$(\boldsymbol{l}_\varepsilon\cdot\mathbf{e}_2) = \frac{\varepsilon_1\sin^2\theta + \varepsilon_3\cos^2\theta}{(\varepsilon_1^2\sin^2\theta + \varepsilon_3\cos^2\theta)^{1/2}} \tag{2.9}$$

It is easy to verify that

$$(\boldsymbol{l}_\varepsilon\cdot\mathbf{e}_1)^2 + (\boldsymbol{l}_\varepsilon\cdot\mathbf{e}_2)^2 + (\boldsymbol{l}_\varepsilon\cdot\boldsymbol{\kappa})^2 = 1$$

3. EVALUATION OF ENERGY LOSS

For a particle possessing spin, the interaction energy has the form

$$e(\boldsymbol{\alpha}\cdot\mathbf{A})$$

where e is the charge on the particle and $\boldsymbol{\alpha}$ is the 4×4 Dirac matrix. Using the method of perturbation (Sokolov, 1958), we find for the probability of radiation the expression

$$w_j = \frac{e^2 c}{2\pi\hbar} \int \Gamma_j\, \delta\left(K - \frac{\omega}{c} - K'\right) d^3\mathbf{q} \tag{3.1}$$

where

$$\begin{aligned}\Gamma_j = {} & \left[\left(1 - \frac{k_0^2}{KK'}\right) - \frac{k \cdot k'}{KK'}\right] \boldsymbol{l}_j^+ \cdot \boldsymbol{l}_j + \frac{2}{KK'}\,[(\boldsymbol{l}_j \cdot \mathbf{k})^2 - (\boldsymbol{l}_j \cdot \mathbf{k})(\boldsymbol{l};\, \mathbf{q})] \\ & + ss'\left[\frac{kk'}{KK'} - \left(1 - \frac{k_0^2}{KK'}\right)\frac{\mathbf{k} \cdot \mathbf{k}'}{kk'}\right] \boldsymbol{l}_j^+ \cdot \boldsymbol{l}_j \\ & + \frac{2}{KK'}\left(1 - \frac{k_0^2}{KK'}\right)[(\boldsymbol{l}_j \cdot \mathbf{k})^2 - (\boldsymbol{l}_j \cdot \mathbf{k})(\boldsymbol{l}_j \cdot \mathbf{q})]\end{aligned}$$

In going from discrete to continuous spectrum, we have used the relation

$$L^{-3} \Sigma \to \frac{1}{8\pi^3} \int d^3\mathbf{q}$$

(Sokolov, 1958).

In (3.1), $\frac{1}{2}\hbar s$ ($s = \pm 1$) is the projection of the spin along the momentum vector of the particle, $\hbar k$, $\hbar c K$ and $\hbar k'$, $\hbar c K'$ are the momentum and energy of the particle respectively before and after the act of radiation, and $c\hbar k_0$ is the rest energy such that $K = (k^2 + k_0^2)^{1/2}$. Also $\mathbf{k}' = \mathbf{k} - \mathbf{q}$.

In problems of this nature, it is usual to consider the particle moving initially either along the optical axis or perpendicular to it.

(i) *Particle initially moves along the optical axis.* When the particle initially moves along the optical axis, $\mathbf{k} = (0, 0, k)$ and the δ-function in (3.1) is independent of φ. Integration with respect to φ merely leads to a factor of 2π. The δ-function is then a function of θ and ω. To integrate with respect to θ, we make the substitution

$$n_j \cos\theta = t_j \tag{3.2}$$

We then use the properties of the δ-function of complex argument noting that for $j = \mu$

$$\frac{q^2\,dq}{\mathcal{M}_\mu} = \frac{1}{2c^3}\frac{\omega}{\varepsilon_1} n_\mu^3\,d\omega$$

$$n_\mu^3\,d(\cos\theta) = \varepsilon_1\mu_3\,dt$$

and for $j = \varepsilon$

$$\frac{q^2\,dq}{\mathcal{M}_\varepsilon} = \frac{1}{2c^3}\frac{\omega}{\mu_1} n_\varepsilon^5\left(\frac{\varepsilon_1^2\sin^2\theta + \varepsilon_3^2\cos^2\theta}{\varepsilon_1^2\varepsilon_3^2}\right)$$

$$n_\varepsilon^3\,d(\cos\theta) = \mu_1\varepsilon_3\,dt$$

After some computation using the above relations and substitution, we find for the energy radiated in unit time

$$W^{\parallel} = W_\mu^{\parallel} + W_\varepsilon^{\parallel(\mathrm{tr})} + W_\varepsilon^{\parallel(\mathrm{long})}$$

$$W_\mu^{\parallel} = \frac{e^2}{c}\int \mathcal{L}\mu_3\left[(\tau_\mu - \beta^{-1}) + \frac{ss'}{\Lambda}\left\{(\tau_\mu - \beta^{-1}) + \frac{\omega\hbar}{cp}(1 - \tau_\mu\beta^{-1})\right\}\right] \times \frac{\omega\,d\omega}{(1 + 4u_\mu r_\mu)^{1/2}}$$

$$W_\varepsilon^{\parallel(\mathrm{tr})} = \frac{e^2}{c}\int \mathcal{L}\mu_1\left[(\tau_\varepsilon - \beta^{-1}) + \frac{ss'}{\Lambda}\left\{(\tau_\varepsilon - \beta^{-1}) + \frac{\omega\hbar}{cp}(1 - \tau_\varepsilon\beta^{-1})\right\}\right] \times \frac{\Delta^{\mathrm{tr}}\omega\,d\omega}{(1 + 4u_\varepsilon r_\varepsilon)^{1/2}} \tag{3.3}$$

$$+ \frac{e^2}{c^2}\sin^2\theta_\varepsilon \int \mu_1\left\{1 + \frac{ss'}{\Lambda}\left(1 - \frac{\omega\hbar}{cp\beta}\right)\right\} \times \frac{\Delta^{\mathrm{tr}}\omega\,d\omega}{(1 + 4u_\varepsilon r_\varepsilon)^{1/2}} \tag{3.4}$$

$$W_\varepsilon^{\parallel(\mathrm{long})} = \frac{e^2}{c}\int \mathcal{L}\mu_1\left[(\tau_\varepsilon - \beta^{-1}) + \frac{ss'}{\Lambda}\left\{(\tau_\varepsilon - \beta^{-1}) + \frac{\omega\hbar}{cp}(1 - \tau_\varepsilon\beta^{-1})\right\}\right. \times \frac{\Delta^{\mathrm{long}}\omega\,d\omega}{(1 + 4u_\varepsilon r_\varepsilon)^{1/2}}$$

$$+ \frac{e^2}{c^2}\cos^2\theta_\varepsilon \int \mu_1\left[\left(1 - \frac{\omega\hbar}{cp\beta}\tau_\varepsilon\right)\left\{1 + \frac{ss'}{\Lambda}\left(1 - \frac{\omega\hbar}{cp\beta}\right)\right\}\right] \tag{3.5}$$

$$\times \frac{\Delta^{\mathrm{long}}\omega\,d\omega}{(1 + 4u_\varepsilon r_\varepsilon)^{1/2}}$$

The following notations have been used:

$$\Delta^{\text{tr}} = 1 + \frac{\varepsilon_3 - \varepsilon_1}{\varepsilon_1}\cos^2\theta_\varepsilon$$

$$\Delta^{\text{long}} = \left(\frac{\varepsilon_3 - \varepsilon_1}{\varepsilon_1}\right)^2 \frac{\varepsilon_1 \sin^2\theta_\varepsilon \cos^2\theta_\varepsilon}{\varepsilon_1 \sin^2\theta_\varepsilon + \varepsilon_3 \cos^2\theta_\varepsilon}$$

$$\mathscr{L} = \frac{\omega\hbar}{2cp}$$

$$\Lambda = \left[1 - \frac{2\omega\hbar}{cp\beta} + \left(\frac{\omega\hbar}{cp}\right)^2\right]^{1/2}$$

$$u_\varepsilon = \frac{\varepsilon_3 - \varepsilon_1}{\varepsilon_1}\mathscr{L}$$

$$r_\varepsilon = \frac{1}{\beta} + \mathscr{L}(\mu_1\varepsilon_3 - 1)$$

$$\cos^2\theta_\varepsilon = \frac{2[(\varepsilon_3 - \varepsilon_1)/\varepsilon_1](\beta^{-1} - \mathscr{L})r_\varepsilon - \mu_1\varepsilon_3[1 - (1 + 4u_\varepsilon r_\varepsilon)^{1/2}]}{2[(\varepsilon_3 - \varepsilon_1)/\varepsilon_1]\{\mu_1\varepsilon_3 - [(\varepsilon_3 - \varepsilon_1)/\varepsilon_1](\beta^{-1} - \mathscr{L})^2\}} \quad (3.6)$$

and τ_j is the positive root of the equation

$$u_j\tau^2 + \tau - r_j = 0$$

For the wave $j = \mu$, make in the above notations where necessary

$$\varepsilon_i \to \mu_i, \qquad \mu_i \to \varepsilon_i \qquad (i = 1, 3)$$

If in (3.6) we expand $(1 + 4u_\varepsilon r_\varepsilon)^{1/2}$, simplify, and put $\varepsilon_1 = \varepsilon_3$ (isotropic case), we obtain

$$\cos^2\theta = \frac{1}{\beta n} + \frac{\omega\hbar}{2cp}(1 - n^{-2}); \qquad n^2 = \varepsilon\mu \quad [\text{see (1.2)}]$$

(ii) *Particle initially moves perpendicular to the optical axis.* In this case $\mathbf{k} = (k_1, k_2, 0)$ and the δ-function now depends on θ and φ. It is more convenient to integrate first with respect to φ. Again we use the properties of the δ-function of complex argument, namely

$$\delta\{f(x)\} = \sum \frac{\delta(x - x_i)}{|f'(x)|}$$

where x_i are the simple roots of the equation $f(x) = 0$. Here

$$f(x) = K - K' - \frac{\omega}{c} = 0$$

or

$$n_j \sin\theta \cos\psi - \frac{1}{\beta} - \mathcal{L}(n_j^2 - 1) = 0 \tag{3.7}$$

and

$$f'(x) = n_j \frac{\omega}{c} k \sin\theta \sin\psi$$

where

$$k \cos\psi = k_i \cos\varphi + k_2 \sin\varphi$$

and

$$k = (k_1^2 + k_2^2)^{1/2}$$

Next we again make the substitution

$$n_j \cos\theta = t_j$$

so that the intensity of radiation takes the form

$$W^{\perp} = W_{\mu}^{\perp} + W_{\varepsilon}^{\perp(\mathrm{tr})} + W_{\varepsilon}^{\perp(\mathrm{long})}$$

where

$$W_{\mu}^{\perp} = \frac{2e^2}{\pi c} \int \frac{\omega\, d\omega}{\Box_{\mu}} \left\{1 + \frac{ss'}{\Lambda}\left(1 - \frac{\omega\hbar}{cp\beta}\right)\right\}$$

$$\times \left[\mu_3 + \mu_3 \mathcal{L}^2(\varepsilon_1\mu_3 - 1) - \mu_1(r_{\mu} + r_0)u_{\mu} - \mu_1 u_{\mu}\mathcal{L}t^2\right.$$

$$\left.\times - \frac{\mu_1 r_0^2}{\varepsilon_1\mu_1 - t^2}\right] \tag{3.8}$$

$$W_{\varepsilon}^{\perp(\mathrm{tr})} = \frac{2e^2}{\pi c} \int \frac{\omega\, d\omega}{\Box_{\varepsilon}} \left\{1 + \frac{ss'}{\Lambda}\left(1 - \frac{\omega\hbar}{cp\beta}\right)\right\}$$

$$\times \left[A_0 + \frac{A_1}{\varepsilon_1\mu_1 - t^2} + \frac{A_2}{\mu_1\varepsilon_1\varepsilon_3 - (\varepsilon_3 - \varepsilon_1)t^2}\right.$$

$$\left.\times + \frac{A_3}{\{\mu_1\varepsilon_1\varepsilon_3 - (\varepsilon_3 - \varepsilon_1)t^2\}^2}\right] \tag{3.9}$$

with

$$A_0 = \mu_1^2\varepsilon_1\mathcal{L}u_\varepsilon$$

$$A_1 = \mu_1 r_0^2$$

$$A_2 = \mu_1^2\varepsilon_1^2\mathcal{L}\left\{(\mu_1\varepsilon_1 - 1)u_\varepsilon - \frac{2}{\beta} + \mathcal{L}\right\} + \mu_1(\varepsilon_3 - \varepsilon_1)r_0^2$$

$$A_3 = -\mu_1^2\varepsilon_1^2\varepsilon_3(\beta^{-1} - \mathcal{L})^2$$

and

$$W_\varepsilon^{\perp(\text{long})} = \frac{2e^2}{\pi c}\int \frac{\omega\, d\omega}{\Box_\varepsilon}\left\{1 + \frac{ss'}{\Lambda}\left(1 - \frac{\omega\hbar}{cp\beta}\right)\right\}$$
$$\times \left[B_0 + B_1 t^2 + \frac{B_2}{\mu_1\varepsilon_1\varepsilon_3 - (\varepsilon_3 - \varepsilon_1)t^2} + \frac{B_3}{\{\mu_1\varepsilon_1\varepsilon_3 - (\varepsilon_3 - \varepsilon_1)t^2\}^2}\right]$$

with

$$B_0 = -\mu_1\mathcal{L}^2 - \frac{1}{\varepsilon_1}(\beta^{-1} - \mathcal{L})^2$$

$$B_1 = -\frac{1}{\varepsilon_1}\mathcal{L}u_\varepsilon$$

$$B_2 = \mu_1^2\varepsilon_1\varepsilon_3\mathcal{L}^2 + (\beta^{-1} - \mathcal{L})^2\mu_1(\varepsilon_1 + \varepsilon_3) \tag{3.10}$$

$$B_3 = -\mu_1^2\varepsilon_1^2\varepsilon_3(\beta^{-1} - \mathcal{L})^2$$

$$r_0 = \beta^{-1} + \mathcal{L}(\varepsilon_1\mu_1 - 1)$$

Further,

$$\Box_\varepsilon = \left[(\mu_1\varepsilon_3 - r_\varepsilon^2) - 2\left(\frac{\varepsilon_3}{2\varepsilon_1} - r_\varepsilon u_\varepsilon\right)t^2 - u_\varepsilon^2 t^4\right]^{1/2}$$
$$= |u_\varepsilon|[(t^2 + a_\varepsilon^2)(b_\varepsilon^2 - t^2)]^{1/2} \tag{3.11}$$

where

$$a_\varepsilon^2 = \frac{\varepsilon_3/2\varepsilon_1 - r_\varepsilon u_\varepsilon}{u_\varepsilon^2}\left[\left\{1 + \frac{u_\varepsilon^2(\varepsilon_3\mu_1 - r_\varepsilon^2)}{(\varepsilon_3/2\varepsilon_1 - r_\varepsilon u_\varepsilon)^2}\right\}^{1/2} + 1\right]$$

$$b_\varepsilon^2 = \frac{\varepsilon_3/2\varepsilon_1 - r_\varepsilon u_\varepsilon}{u_\varepsilon^2}\left[\left\{1 + \frac{u_\varepsilon^2(\varepsilon_3\mu_1 - r_\varepsilon^2)}{(\varepsilon_3/2\varepsilon_1 - r_\varepsilon u_\varepsilon)^2}\right\}^{1/2} - 1\right]$$

For $\Box_\mu$ we make the change $\varepsilon_i \to \mu_i$; $\mu_i \to \varepsilon_i$. The integrals (3.8)–(3.10) are elliptical integrals. Thanks to Byrd and Friendman (1954), we can express them in standard forms. This will give us a clearer picture for analysis. For this purpose we use the substitution

$$\sin\varphi' = \frac{t}{b}\left(\frac{a^2 + b^2}{a^2 + t^2}\right)^{1/2}, \qquad b > t > 0$$

As a consequence of the requirement that the expression under the radical sign in (3.1) be positive, we have complete elliptic integrals. We can therefore represent the intensity of radiation in the following form:

$$W_\mu^\perp = \frac{2e^2}{\pi c}\int \frac{\omega\, d\omega}{|u_\mu|(a_\mu^2 + b_\mu^2)^{1/2}}\left\{1 + \frac{ss'}{\Lambda}\left(1 - \frac{\omega\hbar}{cp\beta}\right)\right\}$$
$$\times\left[\{\mu_3 + \mu_3\mathcal{L}^2(\varepsilon_1\mu_3 - 1) - \mu_1(r_0 + r_\mu)u_\mu\}F(S_\mu)\right.$$
$$- \mu_1\mathcal{L}u_\mu\{(a_\mu^2 + b_\mu^2)E(S_\mu) - a_\mu^2F(S_\mu)\}$$
$$\left.- \frac{\mu_1 r_0^2}{\mu_1\varepsilon_1(\mu_1\varepsilon_1 + a_u^2)}\{a_\mu^2\Pi(\gamma^2, S_\mu) + \mu_1\varepsilon_1F(s_\mu)\}\right] \tag{3.12}$$

$$W_\varepsilon^{\perp(\mathrm{tr})} = \frac{2e^2}{\pi c}\int \frac{\omega\, d\omega}{|u_\varepsilon|(a_\varepsilon^2 + b_\varepsilon^2)^{1/2}}\left\{1 + \frac{ss'}{\Lambda}\left(1 - \frac{\omega\hbar}{cp\beta}\right)\right\}$$
$$\times\left[A_0F(s_\varepsilon) + \frac{A_1}{\varepsilon_1\mu_1(\varepsilon_1\mu_1 + a_\varepsilon^2)}\{a_\varepsilon^2\Pi(\alpha_0^2,\ s_\varepsilon) + \varepsilon_1\mu_1F(s_\varepsilon)\}\right.$$
$$+ \frac{A_2}{\mu_1\varepsilon_1\varepsilon_3(\mu_1\varepsilon_1\varepsilon_3 + (\varepsilon_3 - \varepsilon_1)a_\varepsilon^2)}\{(\varepsilon_3 - \varepsilon_1)a_\varepsilon^2\Pi(\alpha_1^2, s_\mu)$$
$$+ \mu_1\varepsilon_1\varepsilon_3F(s_\varepsilon)\}$$
$$+ A_3\left(\frac{s_\varepsilon^2}{\mu_1\varepsilon_1\varepsilon_3\alpha_1^2}\right)^2\left\{\frac{s_\varepsilon^4 - \alpha_1^4 - 2\alpha_1^2 s_\varepsilon^2 s_\varepsilon'^2}{2(1 - \alpha_1^2)s_\varepsilon^4}F(s_\varepsilon) + \frac{(\alpha_1^2 - s_\varepsilon^2)\alpha_1^2}{2(1 - \alpha_1^2)s_\varepsilon^2}E(s_\varepsilon)\right.$$
$$\left.\left.+ \frac{(\alpha_1^2 - s_\varepsilon^2)(s^2 - 2\alpha_1^2 s^2 + 2\alpha_1^2 - \alpha_1^4)}{2(1 - \alpha_1^2)s_\varepsilon^4}\Pi(\alpha_1^2, s_\varepsilon)\right\}\right] \tag{3.13}$$

$$W_\varepsilon^{\perp(\text{long})} = \frac{2e^2}{\pi c}\int \frac{\omega\, d\omega}{|u_\varepsilon|(a_\varepsilon^2 + b_\varepsilon^2)^{1/2}}\left\{1 + \frac{ss'}{\Lambda}\left(1 - \frac{\omega\hbar}{cp\beta}\right)\right\}$$

$$\times \Bigg[B_0 + B_1\{(a_\varepsilon^2 + b_\varepsilon^2)E(s_\varepsilon) - a_\varepsilon^2 F(s_\varepsilon)\}$$

$$+ \frac{B_2}{\mu_1\varepsilon_1\varepsilon_3(\mu_1\varepsilon_1\varepsilon_3 + (\varepsilon_3 - \varepsilon_1)a_\varepsilon^2)}$$

$$\times \{(\varepsilon_3 - \varepsilon_1)a_\varepsilon^2\Pi(\alpha_1^2, s_\varepsilon) + \mu_1\varepsilon_1\varepsilon_3 F(s_\varepsilon)\}$$

$$+ \frac{B_3}{2(1 - \alpha_1^2)(\mu_1\varepsilon_1\varepsilon_3\alpha_1^2)^2}\{(s_\varepsilon^2 - \alpha_1^4 - 2\alpha_1^2 s_\varepsilon^2 s_\varepsilon'^2)F(s_\varepsilon)$$

$$+ \alpha_1^2(\alpha_1^2 - s_\varepsilon^2)E(s_\varepsilon)$$

$$+ (\alpha_1^2 - s_\varepsilon^2)(s_\varepsilon^2 - 2\alpha_1^2 s_\varepsilon^2 s_\varepsilon^2 + 2\alpha_1^2 - \alpha_1^4)\Pi(\alpha_1^2, s_\varepsilon)\}\Bigg] \tag{3.14}$$

$s'^2 = 1 - s^2$, $F(s_\varepsilon)$, $E(s)$, and $\Pi(\alpha, s)$ are the complete elliptic integrals of the first, second, and third kinds, respectively. In the foregoing expressions, the following notations have further been used:

$$s_j = \frac{b_j}{(a_j^2 + b_j^2)^{1/2}}$$

$$\gamma^2 = \frac{(\mu_1\varepsilon_1 + a_\mu^2)s_\mu^2}{\mu_1\varepsilon_1}$$

$$\alpha_0^2 = \frac{(\mu_1\varepsilon_1 + a_\varepsilon^2)s_\varepsilon^2}{\mu_1\varepsilon_1}$$

$$\alpha_1^2 = \frac{\{\mu_1\varepsilon_1\varepsilon_3 + (\varepsilon_3 - \varepsilon_1)a_\varepsilon^2\}s_\varepsilon^2}{\mu_1\varepsilon_1\varepsilon_3}$$

4. CONCLUSION

The field of radiation has been resolved into its transverse and longitudinal components. The energy loss through radiation therefore reflects the contributions from these two components. We find in taking the limits $\varepsilon_1 = \varepsilon_3 = \varepsilon$ and $\mu_1 = \mu_3 = \mu$ that $s_j = \alpha_j = 0$, so that we can use the properties of the elliptic integrals

$$F(\pi/2, 0) = E(\pi/2, 0) = \Pi(\pi/2, 0, 0) = \pi/2$$

We then obtain for the isotropic limit

$$W_{\varepsilon}^{\perp(\text{long})} = 0$$

as should be expected

$$W^{\perp(\text{tr})} = W_{\mu}^{\perp(\text{tr})} + W_{\varepsilon}^{\perp(\text{tr})} \qquad (n^2 = \varepsilon\mu)$$

$$= \frac{e^2}{c}\int \omega\, d\omega \left[1 - \frac{1}{\beta^2 n^2} - \frac{\omega\hbar}{cp\beta}(1 - n^{-2}) + \left(\frac{n\omega\hbar}{2cp}\right)^2 (1 - n^{-4})\right]$$

This agrees with Sokolov's result [see above, equation (1.3)], except for a factor β. The above result confirms our earlier assertion that the whole energy loss cannot be attributed to the Cerenkov effect alone.

We also stated in the introduction that focus will be on the role of the spin. Examination of the expressions (3.3)–(3.5) and (3.12)–(3.14) shows that this role is governed by the quantity Λ. Whether or not the spin plays any role depends upon Λ being real, that is, if

$$1 > \frac{2\omega t}{cp}\left(\beta^{-1} - \frac{\omega\hbar}{2cp}\right)$$

or

$$\frac{\omega\hbar}{cp}\left(\beta^{-1} - \frac{\omega\hbar}{2cp}\right) < \frac{1}{2}$$

or

$$\frac{\omega\hbar}{cp} < \frac{1}{\beta}\{1 \mp (1 - \beta^2)^{1/2}\} \tag{4.1}$$

When this condition is satisfied, the expressions for the energy show that the role of the spin can lead to an increase or decrease of radiation determined by the quantity

$$1 - \frac{\omega\hbar}{cp\beta}$$

Thus, if $\beta = v/c > \omega\hbar/cp$, the intensity increases when there is no spin flip. If, however, radiation is accompanied by spin flip, the intensity decreases.

But if $\beta < \omega\hbar/cp$ the reverse occurs; but note (4.1). Physically, we note that

$$\frac{\omega\hbar}{cp} = \frac{\text{energy of photon}}{\text{kinetic energy of particle}} \tag{4.2}$$

Thus whether an increase or decrease in energy loss occurs as a result of the spin behavior will depend on the ratio of (4.2) in relation to v/c:

$$\frac{v}{c} \gtrless \frac{\text{energy of photon}}{\text{KE of particle}} \qquad \begin{cases} \text{no spin flip} \\ \text{spin ignored} \\ \text{spin flip} \end{cases}$$

Further, we must ensure that the expression for the intensity of radiation is real. This is met by the requirement that the expression under the radical sign in (3.11) is positive and this will be satisfied if $b_j > 0$, which gives

$$\varepsilon_3\mu_1 - r_\varepsilon^2 > 0 \qquad \text{or} \qquad \varepsilon_3\mu_1 - \left\{\beta^{-1} + \frac{\omega\hbar}{2cp}(\mu_1\varepsilon_3 - 1)\right\}^2 > 0$$
$$(j = \varepsilon)$$
$$\varepsilon_1\mu_3 - r_\mu^2 > 0 \qquad \text{or} \qquad \varepsilon_1\mu_3 - \left\{\beta^{-1} + \frac{\omega\hbar}{2cp}(\varepsilon_1\mu_3 - 1)\right\}^2 > 0$$
$$(j = \mu) \tag{4.3}$$

For an isotropic medium this reduces to

$$1 - \left\{\beta^{-1}n^{-1} + \frac{n\omega\hbar}{2cp}(1 - n^{-2})\right\}^2 > 0 \tag{4.4}$$

and reduces to the classical condition by Tamm and Frank when $\hbar \to 0$. Thus (4.3) and (4.4) provide us with equivalent conditions for the radiation of the corresponding waves in anisotropic media.

Finally, we see that the pattern of radiation when the particle initially moves along the optical axis is similar to what obtains in the case of an isotropic medium. Here as in the case of an isotropic medium there is a well-defined cone of radiation. There is no such well-defined cone when the particle initially moves perpendicular to the optical axis. This is because the cone now depends on both θ and φ. This is also borne out by the fact that in the classical case only one wave is radiated when the particle initially moves along the optical axis (Ginsburg, 1940a,b; Zrelov, 1968). If in (3.3) we put $\hbar = 0$, then $W_\mu^\parallel = 0$. Thus only the wave $j = \varepsilon$ contributes to the radiation.

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